On 33-hole solitaire positions with rotational symmetry

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When writing *The ins and outs of peg solitaire*, I concluded the chapters on the standard English 33-hole board with the observation that no solution to the "central game" (start by vacating the central hole, and play to finish with a single man in this hole) could pass through an intermediate position having four-fold rotational symmetry. It recently occurred to me to try to find solutions which passed through intermediate positions having just two-fold rotational symmetry, and to my surprise I found that these did not appear to exist either. The analysis given below shows why not. It uses John Conway's "balance sheet", which is explained in chapter 23 of *Winning ways for your mathematical plays* (E. L. Berlekamp, J. H. Conway, and R. K. Guy, second edition 2004) and in chapter 6 of *The ins and outs of peg solitaire*, but to make the presentation complete we start by summarizing the essentials (readers already familiar with the balance sheet may skip this next section). We use the additive notation of *The ins and outs*.

Conway's balance sheet

We label the columns of the board *a*...*g* from left to right and the rows 1...7 from top to bottom (chess players are asked to note that in solitaire we put row 1 at the top), and assign values to the holes as below:

	a	*b*	*c*	*d*	*e*	*f*	*g*
1			$-b$	β	$-b+\beta$		
2			b	$a+\beta$	$b+\beta$		
3	$-a$	a	0	a	0	a	$-a$
4	α	$b+\alpha$	b	$c+\alpha+\beta$	$b+\beta$	$b+\alpha+\beta$	α
5	$-a+\alpha$	$a+\alpha$	0	$a+\alpha$	0	$a+\alpha$	$-a+\alpha$
6			b	$a+\alpha+\beta$	$b+\beta$		
7			$-b$	β	$-b+\beta$		

The values *a*, *b*, *c* in the diagram are distinct from the column labels *a*, *b*, *c*, but in practice this does not cause confusion because we use the latter only in conjunction with a row number (thus *a*3, *c*1, etc). The diagram gives the values of the individual holes, and to obtain the value of a position we add the values of the occupied holes, subject to the condition that $\alpha + \alpha = \beta + \beta = 0$ (in other words, all we are interested in the cases of α and β are their parities). Thus in the central game, the value of the initial position (every hole occupied apart from the centre) turns out to be $4a + 4b$ (the alphas and betas cancel out), while the value of the target position is clearly $c + \alpha + \beta$.

Now let us consider the effect of a move on the value of a position. If this move takes a man from hole X over hole Y and into hole Z, we lose the values of holes X and Y, but gain that of hole Z. For example, if we start a solution to the central game by playing *d*2-*d*4, we reduce the initial value $4a + 4b$ of the position by $a + \beta$ (hole *d*2) and a (hole *d*3) but increase it by $c + \alpha + \beta$ (hole *d*4), resulting in a new value $2a + 4b + c + \alpha$. To put it another way, this move has decreased the value of the position by $2a - c + \alpha$.

If we consider each of the possible moves on a solitaire board in this way, we obtain the table overleaf.

Move	Resulting decrease in value of position
Horizontal move into $a3$, $g3$, $a5$, or $g5$	$2a$
Vertical move into $d1$ or $d7$	$2a$
Horizontal move across $d2$ or $d3$	a
Horizontal move across $d5$ or $d6$	$a + \alpha$
Vertical move into $c1$, $e1$, $c7$, or $e7$	$2b$
Horizontal move into $a4$ or $g4$	$2b$
Vertical move across $b4$ or $c4$	b
Vertical move across $e4$ or $f4$	$b + \beta$
Horizontal move across the centre	$c + \alpha$
Vertical move out of the centre	$c + \alpha$
Vertical move across the centre	$c + \beta$
Horizontal move out of the centre	$c + \beta$
Vertical move into the centre	$2a - c + \alpha$
Horizontal move into the centre	$2b - c + \beta$
Any other move	0

We call the quantities a, $a + \alpha$, b, $b + \beta$, $c + \alpha$, $c + \beta$, $2a - c + \alpha$, and $2b - c + \beta$ the *primitives* of the balance sheet. In principle, we could also count $2a$ and $2b$ as primitives, but in practice we don't because they can be expressed as sums of other primitives; for example, $2a = (a) + (a) = (a + \alpha) + (a + \alpha) = (c + \alpha) + (2a - c + \alpha)$.

Now the effect of playing a sequence of moves is to decrease the value of the position by the sum of the associated primitives, so *a necessary condition for a problem to be solvable is that the difference between the values of the initial and final positions be expressible as a sum of primitives*. If this difference cannot be so expressed, we can say without further analysis that the problem cannot be solved.

It is therefore useful to know what quantities cannot be expressed as sums of primitives. There turn out to be three classes.

(a) Quantities with a negative a or b component. Since no individual primitive has a negative a or b component, no sum of them can have such a component.

(b) Certain quantities with a negative c component. None is relevant to present purposes, and we shall not examine them further.

(c) The quantities

α		$b + \alpha$		$2b + \alpha$	c
β	$a + \beta$		$2a + \beta$		
$\alpha + \beta$	$a + \alpha + \beta$	$b + \alpha + \beta$			$c + \alpha + \beta$

The quantities in this third class arise because the only primitives involving α also involve a or c, and the only primitives involving β also involve b or c. For example, consider $2b + \alpha$. We need a primitive containing α, and since we don't have an a component the only candidate is $(c + \alpha)$. Subtracing this from $2b + \alpha$ leaves $2b - c$, and the only primitive containing a negative c component and not containing a is $(2b - c + \beta)$. Subtracting this in turn leaves β, which is inexpressible. Similar arguments apply to the values which involve β.

Applying the balance sheet to positions with two-fold rotational symmetry

A position with two-fold rotational symmetry includes both of $c1$/$e7$ or neither of them, both of $d1$/$d7$ or neither of them, and so on. If we evaluate these pairs according to the balance sheet, we obtain the values overleaf.

Pair	Value
$c1/e7$, $e1/c7$	$-2b + \beta$
$d1/d7$	0
$c2/e6$, $e2/c6$	$2b + \beta$
$d2/d6$	$2a + \alpha$
$a3/g5$, $g3/a5$	$-2a + \alpha$
$b3/f5$, $d3/d5$, $f3/b5$	$2a + \alpha$
$c3/e5$, $e3/c5$	0
$a4/g4$	0
$b4/f4$, $c4/e4$	$2b + \beta$

Since each $2a$ (positive or negative) is accompanied by an α, the pairs involving a make an overall contribution $-4a$, $-2a + \alpha$, 0, $2a + \alpha$, $4a$, $6a + \alpha$, or $8a$ to the value of the position according as to how they are occupied, and similarly the pairs involving b make an overall contribution $-4b$, $-2b + \beta$, 0, $2b + \beta$, $4b$, $6b + \beta$, or $8b$. Additionally, there is a contribution $c + \alpha + \beta$ if the centre is occupied.

Now let us look at solutions to the central game. Without loss of generality, we can suppose the first move of the solution to be *d*2-*d*4, and as we have seen this reduces the value of the position (initially $4a + 4b$) to $2a + 4b + c + \alpha$. Furthermore, the final move of the solution must be *d*2-*d*4, *b*4-*d*4, *f*4-*d*4, or *d*6-*d*4, so the holes occupied immediately before this final move must be *d*2/*d*3, *b*4/*c*4, *e*4/*f*4, or *d*5/*d*6, and these give positions with values $2a + \beta$, $2b + \alpha$, $2b + \alpha$, and $2a + \beta$ respectively. So if we are to have a position with rotational symmetry along the way, this position must be reachable from a position with value $2a + 4b + c + \alpha$, and a position with value $2a + \beta$ or $2b + \alpha$ must be reachable from it. This immediately constrains us to positions whose values have an a component not greater than $2a$ and a b component not greater than $4b$, else they will not be reachable from a position with value $2a + 4b + c + \alpha$, and whose values have non-negative a and b components, else it will not be possible to reach a position with value $2a + \beta$ or $2b + \alpha$ from them. Furthermore, the a and b components cannot both be zero, else again it will not be possible to reach a position with value $2a + \beta$ or $2b + \alpha$.*

All this reduces us to the ten cases shown under "Value of position" in the following table.

From $2a + 4b + c + \alpha$		Value of position	To $2a + \beta$		To $2b + \alpha$	
$\alpha + \beta$	×	$2a + 4b + c + \beta$				
$2b + \alpha$	×	$2a + 2b + c$				
$4b + \alpha + \beta$		$2a + c + \beta$	c	×	$-2b$ ...	×
$2a + \beta$	×	$4b + c + \alpha + \beta$				
$2a + 2b$		$2b + c + \alpha$	$-2a$ ...	×	c	×
c	×	$2a + 4b + \alpha$				
$2b + c + \beta$		$2a + 2b + \alpha + \beta$	$2b + \alpha$	×	$2a + \beta$	×
$4b + c$		$2a + \alpha$	$\alpha + \beta$	×	$-2b$ ...	×
$2a + c + \alpha$		$4b$	$-2a$ ...	×	$2b + \alpha$	×
$2a + 2b + c + \alpha + \beta$		$2b + \beta$	$-2a$ ...	×	$\alpha + \beta$	×

In each case, the left-hand column shows the difference in value between the position after the assumed opening move *d*2-*d*4 and the candidate position, and we see that in four cases this cannot be expressed as a sum of primitives (in other words, no position with this value is reachable). In the remaining six cases, the two columns to the right show the differences in value between the candidate position and the alternative target positions, and in no case is one of the target positions reachable.

So no solution to the central game can pass through a position having even two-fold rotational symmetry.

* Readers who have not previously encountered balance sheet analyses are sometimes puzzled to see positions with values such as $2b + \alpha$, since we have shown $2b + \alpha$ to be inexpressible as a sum of primitives. However, there is no contradiction. A quantity such as $2b + \alpha$ is perfectly acceptable as the value of a *position*. What it cannot be is the *difference* between the values of two separate positions, one of which can be reached from the other by a sequence of legal moves.

Some observations

The analysis given here could have been presented in more elementary terms, but the exposition would have been much longer and would have involved a great deal more case-by-case examination. The result could also have been "proved" by computer, by getting it to enumerate every solution to the central game and showing that none passed through a rotationally symmetrical position. However, computer "proofs" of non-existence by allegedly exhaustive search and failure to find are never entirely satisfactory, because there is always the possibility that a machine or program error may have caused a batch of candidates to be overlooked. It is much more satisfactory, logically and intellectually, to have an argument, laid out step by step for independent checking by anybody who feels inclined, that the existence of a such-and-such must lead to a contradiction somewhere or other.

In the analysis as presented here, the use of the balance sheet reduced to ten the number of separate cases that had to be considered. This number could have been reduced still further by observing that if a solution passing through a position with rotational symmetry should exist then a solution passing through such a position with the centre occupied must also exist (since if the position in the solution which we have found has the centre unoccupied, playing the jumps in reverse order will give a solution in which the corresponding position has the centre occupied). This would have allowed the omission of the second five cases from the ten-case table on page 3. However, I decided that it would be simpler for the reader, at least on a first reading, to present all ten cases than to add a separate explanation that only half of them were necessary.

When he first sent me the balance sheet, back in the summer of 1963, John Conway remarked that he had been trying to reduce solitaire to a triviality (it bound into one a number of separate arguments that previously had had to be applied sequentially). The balance sheet doesn't quite do this, but it vastly increases the number of problems that can be resolved with any given expenditure of time and effort, and by so doing it has made a very great contribution to the development of the game.

In recent years, I have been doing other things, and I have not been keeping up with the solitaire literature. It is therefore perfectly possible that this result has already been discovered and published. If this has happened, please let some reader bring the matter to my attention.