Sphere Octahedron Puzzles

by George Bell

A fascinating feature of face-centered cubic (FCC) sphere packings is a wide variety of shapes that can be formed: tetrahedrons, cubes, and octahedrons. **Polyspheres** are puzzle pieces made by joining spheres packed in the FCC lattice. In CFF 81, we came up with a classification scheme for polyspheres, and considered puzzles in the shape of a tetrahedron [1]. Figure 1a shows an octahedron made from 19 spheres. Other size octahedrons are possible, of course, but this is a nice size for a puzzle.

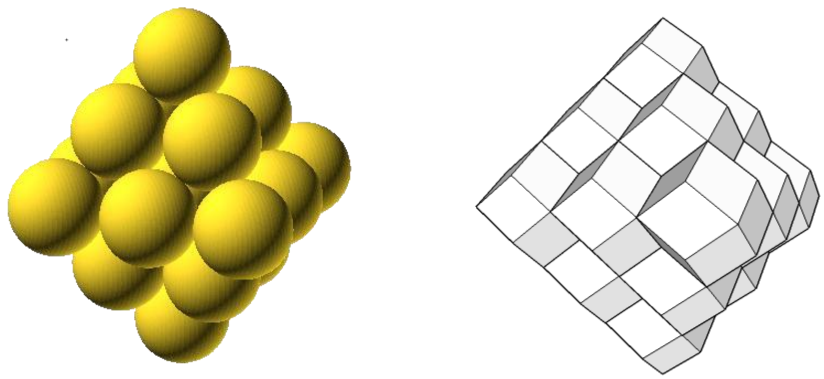

Figure 1. (a) An octahedron made from 19 spheres, or (b) 19 RD.

The problem with octahedrons built from spheres is that they tend to fall apart (when sitting on a table, one face is overhanging). For this reason only the top half is often considered—the square-base pyramid. But in this article we will stick with the more symmetric octahedron, and look for puzzles that are interlocking.

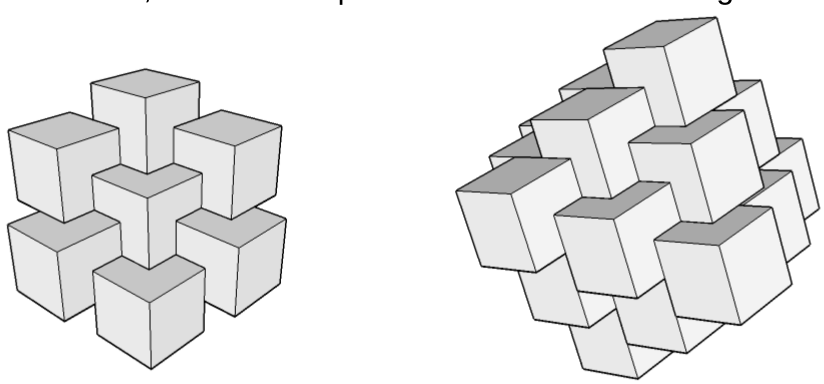

Figure 2. (a) A hexagonal layer, (b) an octahedron made from 19 cubes.

Instead of spheres, we can alternatively make our pieces from rhombic dodecahedrons (RD), as in Figure 1b. We can also use cubes by packing them in a special way. Figure 2a shows six cubes surrounding a central cube, where the cubes all touch one another along quarter faces. This packing can be extended over the entire plane, and is geometrically the same as hexagonal sphere packing (as in a honeycomb). We can add layers above and below, as in FCC sphere packing, with cubes in different layers touching along half faces.   We call this cube-packing scheme “Weird Tilted Cube” (WTC) geometry. The most well-known puzzle which uses this confusing geometry is Stewart Coffin’s **Three Piece Block** [2, p. 57-58].

Figure 2b shows an octahedron made from 19 cubes in the WTC geometry. In what follows, all puzzle designs can be built from spheres, RDs, or cubes in the WTC geometry. Although the underlying packing is always FCC, whether the puzzle can be assembled, or is interlocking can vary depending on the components.

The WTC geometry has one feature which complicates matters—it has lost some of the symmetry of the FCC lattice. The loss of symmetry can be seen in the unique plane where all the cubes touch along quarter faces (Figure 2a). A puzzle design can be realized in WTC in different ways depending on how this plane is oriented relative to the octahedron. For this reason we won’t give results for each puzzle in the WTC geometry. But keep in mind that all polysphere puzzles can be realized in multiple ways in this strange geometry.

Informally, a puzzle is **interlocking** if the pieces hold one another together in the final shape. It can be quite difficult to determine if a puzzle made from polyspheres is interlocking, and if an interlocking puzzle can be assembled. Imagine the pieces are rigid and geometrically exact—under these conditions very few interlocking sphere puzzles can be assembled. Often a slight amount of flexibility is required for assembly. Alternatively, we can keep the pieces rigid, but undersize the spheres, keeping them the same distance apart[1]. Of course, if the pieces are flexible enough, or we undersize the spheres enough, almost any puzzle can be assembled. So there is a gray area of puzzles that are interlocking and can be assembled given slightly flexible or undersized pieces. The amount of flexibility required varies with the puzzle. Some of the designs below have been made out of wood or metal, others only seem to work if made from sufficiently flexible plastic.

19-Sphere Octahedron Puzzles

In what follows we present a number of interlocking octahedron designs. All were originally designed using spheres, but some can be assembled using RD or cubes, and may interlock better made from these components. Table 1 introduces a numerical code for interlock and assembly type.

Puzzle interlock code:
0 ⇨ not interlocking (and can be assembled).
1 ⇨ interlocking and can be assembled with rigid, exact size pieces.
2 ⇨ interlocking and can be assembled with slightly flexible or undersized pieces.
3 ⇨ interlocking and can be assembled using coordinate motion.
4 ⇨ interlocking but cannot be assembled.

Table 1. Puzzle interlock codes.

[1] Woodworkers can use spheres of the correct size, but lengthen slightly the dowels joining them.

In CFF 81 [1], I found that a good way to search for interlocking polysphere puzzles was to search for all designs with identical or mirror image pieces. BurrTools [3] was used to carry out this search. Identical pieces may make the puzzle easier, but it seems to give a better chance that the puzzle will interlock. In what follows we will see that the restriction of identical pieces uncovers coordinate motion designs.

The 19-sphere octahedron cannot be divided into identical polyspheres as it contains a prime number of spheres. If we remove the centre sphere, however, it can be divided into three identical 6-sphere pieces (hexaspheres). The other possibility we consider is to keep the centre sphere and divide the octahedron into three identical or mirror image pentaspheres plus one tetrasphere.

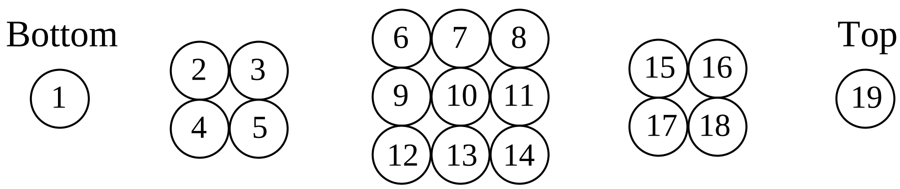

Figure 3. Level diagram numbering the spheres in a 19-sphere octahedron.

Piece	Type	Sphere numbers
4N4 (4-sphere tetrahedron)	Symmetrical, non-planar	2, 3, 7, 10
4X2 (Y tetrahex, Figs 4 & 5)	Planar but chiral from RD	4, 10, 11, 15
4X3	Planar but chiral from RD	2, 6, 7, 8
4M2	Chiral	3, 5, 6, 7
5S8 (U pentomino, Figure 5)	Planar	6, 7, 8, 9, 11
5N2 (Figure 6)	Symmetrical, non-planar	2, 3, 5, 6, 14
5M16 (Figure 4)	Chiral	3, 5, 6, 7, 14
5M44	Chiral	2, 3, 5, 6, 8
5M58	Chiral	4, 6, 9, 10, 11
5M60	Chiral	2, 6, 7, 8, 11
5M62	Chiral	2, 3, 6, 8, 16
6M726 (5N2 plus 5M16)	Chiral	2, 3, 5, 6, 7, 14
6M864 (5S8 plus one)	Chiral	3, 6, 7, 8, 9, 11

Table 2. Puzzle pieces used in the 19-sphere octahedron.

Figure 3 shows our numbering of the spheres in the octahedron, and Table 2 describes the pieces used by puzzles in this article. For a detailed explanation of the piece notation, see CFF 81 [1]. A **Chiral** piece is one that is not identical to its mirror image, these are denoted by type “M” (and “X” made from RD). If so the mirror image piece is denoted by the same piece number with the lower case m (or x).

Of particular interest is the piece **5M16** (Figure 4). In this piece the sphere centres lie on a helix. This can be seen most easily by rotating the piece so that it extends vertically, it then contains one sphere in each layer in a rotating pattern: 1, 2, 7, 16, 19. As an aside, we note that there are *two* distinct types of helical polyspheres: those like **5M16** (and **4M2**) where the axis of the helix is perpendicular to a square packing plane, and another (**5M5** and **4M1**) where the axis is perpendicular to a hexagonal packing plane. The second type of helical piece I use in my **Screwy Cube** puzzle [4].

The **Screwy Octahedron** comes in several variations which all use the screw shaped piece **5M16,** and possibly its mirror image **5m16**. The original version was my IPP30 exchange puzzle, made in plastic using 3D printing [4], the pieces must have some flexibility to assemble. I do not suggest making this puzzle out of wood (it will break). I eventually discovered that the version with three identical copies of **5M16** assembles from rigid pieces using coordinate motion, and made this puzzle out of stainless steel (Figure 4). The third, most difficult variation uses four different pieces, and requires flexibility to assemble.

Puzzle name (designer)	Pieces	Interlock code with Spheres	Interlock code with RD
Ball Octahedron (Coffin #232)	4m2, 5m44, 5M58, 5M16	2	4
Screwy Octahedron original	4x2, 2 × 5M16, 5m16	2	4
Screwy Octahedron CM	4x2, 3 × 5M16	3	4
Screwy Octahedron all different	4x3, 5M16, 5m16, 5N2	2	4
U'y Octahedron or Octetra (Genel)	4X2, 3 × 5S8	3	4
U'y Octahedron, 3 identical pieces	3 × 6M864	3	4
Octahedron with Child	4N4, 3 × 5N2	1	3
Octahedron with Child variation	4N4, 3 × 5M60	0	3
Tricky Octahedron	4x2, 3 × 5M62	2	4
3 Piece Octahedron (Pontalti & Bell)	3 × 6M726	2	4

Table 3. Summary of 19-sphere octahedron puzzles (see Table 1 for the interlock codes). If a designer is not listed the puzzle is due to the author.

Figure 4. Screwy Octahedron CM, made from stainless steel using 3D printing [4].

A simple and elegant design I call **U'y Octahedron**, in reference to the shape of the two kinds of planar pieces. This puzzle has been made out of wood by Tom Lensch [5] (Figure 5), and requires coordinate motion to assemble. Later, I found out this design was discovered over 10 years ago by Viktor Genel as the first part of a puzzle he calls **Octetra** [6], although it was not widely distributed.

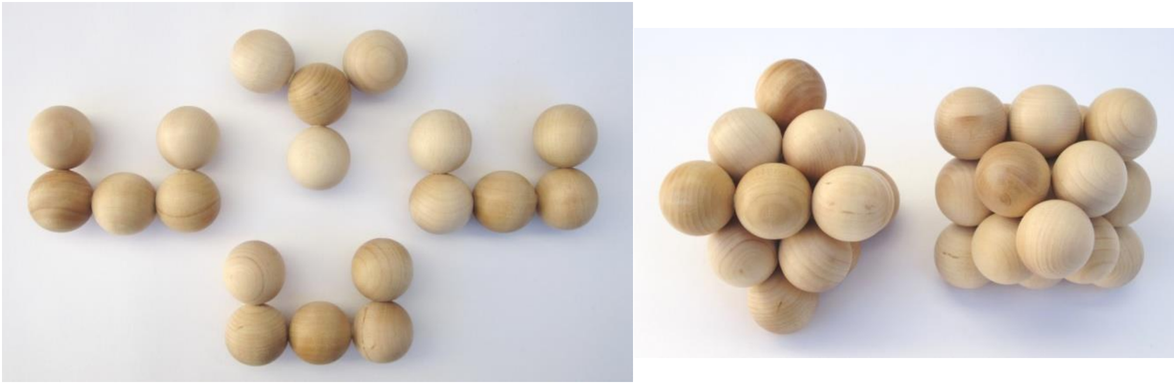

Figure 5. U'y Octahedron, made by Tom Lensch.

ieres, RD, or cubes. Common to both versions is the "child" locking piece, a 4-sphere tetrahedron (**4N4**). Made from spheres, these puzzles are very loosely interlocking (if at all) and seem to require piece rotation to assemble. The puzzle interlocks solidly made from RD, and assembles using some form of coordinate motion. Figure 6 shows a wood version made by John Devost using truncated RD (edge-beveled cubes).

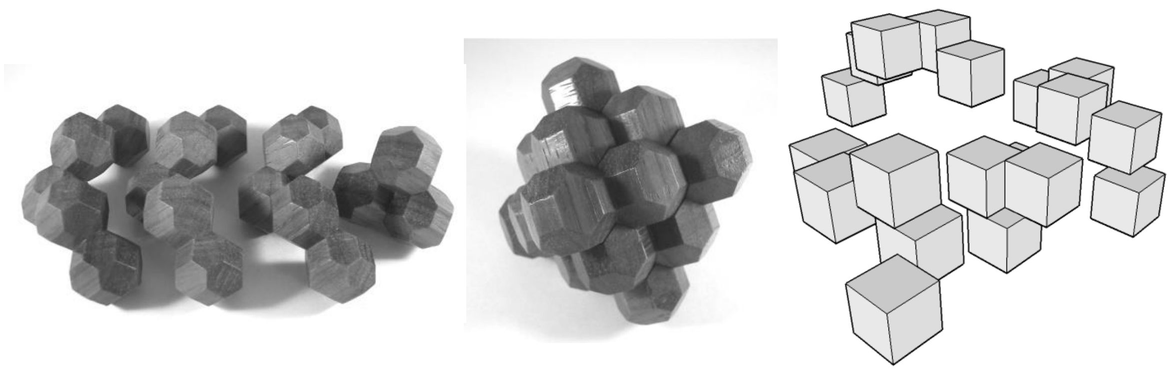

Figure 6. Octahedron with Child, made by John Devost.

Figure 6c shows the puzzle pieces in the WTC geometry [7]. Because of the loss of symmetry in this geometry, all the pieces are different. The puzzle now assembles one piece at a time (it no longer requires coordinate motion). The last piece (the tetrahedron) slides into place with three moves. By changing the orientation of the "special plane" relative to the octahedron, several other versions of the puzzle can be created in the WTC geometry—not all can be assembled.

So far we have considered only connected polysphere pieces. We could also consider joining spheres that do not touch. For example, spheres vertically aligned in different square packing layers, such as spheres 1 and 10 or 2 and 15 in Figure 3. The physical connection can be made using a longer cylinder. Leonard Gordon calls these "**contraplanar** pieces" [8]. An interlocking octahedron puzzle which uses such pieces is Rolando Pontalti's **18+1 Octahedron** [9] (made from truncated RD). This puzzle uses six identical pieces formed by joining spheres 1, 2 and 15 in our notation, and is closely related to the standard **Diagonal Burr** [2, p. 81-86]. Interlocking tetrahedron puzzles utilizing contraplanar pieces are Viktor Genel's **Octetra** [6], and Markus Götz's **Curse of the Pharaoh** [10].

References

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