Five Intersecting Tetrahedra

by George Bell

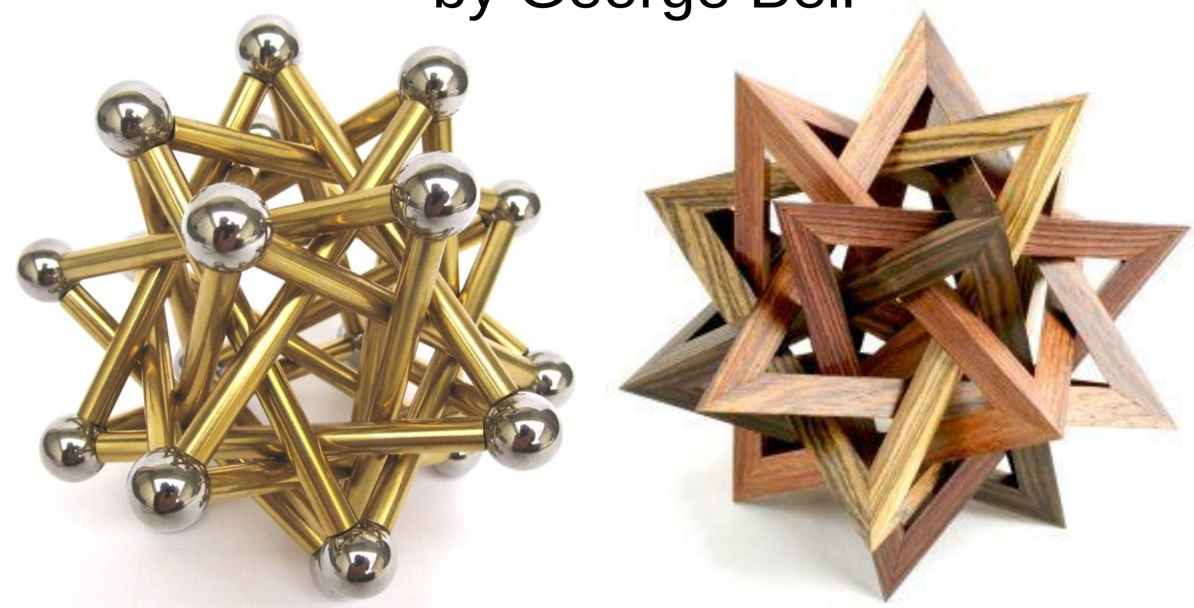

Figure 1. Two versions of Five Intersecting Tetrahedra.

Figure 1a shows a beautiful geometrical object composed of five self-intersecting tetrahedra, made by John Devost [1]. It is made from 30 identical solid brass cylinders with magnets embedded in the ends, together with 20 ball bearings. When assembled, the ball bearing centres lie at the vertices of a dodecahedron, and on each face the rods show a nice spiral form. Some call the assembly a puzzle, others say it is only a geometric construction. I can attest that assembly is not easy!

In Figure 1b the same geometrical shape (actually its mirror image) is made from 30 identical wood struts of triangular cross-section with embedded magnets. This puzzle “Merkaba” is exquisitely crafted in exotic hardwoods by Lee Krasnow [2].

The cylinder or rod version is easier to make, and inexpensive copies can be made from colored pencils. If you wish to build one, at what length do you cut your colored pencils? If you cut them too long, the structure will be loose, but if you cut them too short, it won’t go together. The perfect length can be determined by trial and error, or by looking for intersecting solids in a CAD program, but here I will calculate it exactly. I present these calculations with the hope that these methods will prove useful in other similar puzzles.

Five Intersecting Tetrahedra made from rods and ball bearings

A simple representation of a tetrahedron uses the 4 vertices of a cube which are separated by a face diagonal: (1,-1,-1), (-1,1,-1), (-1,-1,1) and (1,1,1). Any pair is separated by $2\sqrt{2}$, the edge length of the tetrahedron. We now rotate this tetrahedron about a certain axis in multiples of 360/5 = 72 degrees to obtain the other four tetrahedra in the structure. This rotation is difficult to conceptualize because the rotation axis is not aligned with any edge of the tetrahedron or underlying cube, but is dictated by the geometry of a dodecahedron.

Indeed, it is exactly this subtlety that makes the object tricky to assemble. Mathematically, this rotation is multiplication by the “golden rotation matrix”:

$$R = \frac{1}{2}\begin{bmatrix} 1 & -\varphi & 1/\varphi \\ \varphi & 1/\varphi & -1 \\ 1/\varphi & 1 & \varphi \end{bmatrix}$$

Here $\varphi = (\sqrt{5}+1)/2$ is the golden ratio, which satisfies the equation $\varphi^2 = \varphi + 1$, as well as $1/\varphi = \varphi - 1$.

The diameter d of the thickest rod which can be used is the minimum value of the distance between any pair of edges among different tetrahedra. This distance is attained between the edge from $\vec{x}_1 = (-1,-1,1)$ to $\vec{x}_2 = (1,1,1)$ and the edge from $\vec{x}_3 = R\vec{x}_1 = (1/\varphi, -\varphi, 0)$ to $\vec{x}_4 = R\vec{x}_2 = (0, 1/\varphi, \varphi)$. To calculate the minimum distance between these skew lines, we use a standard vector projection formula, $d = \left|\vec{c}\cdot(\vec{a}\times\vec{b})\right| / \left|\vec{a}\times\vec{b}\right|$, where $\vec{a} = \vec{x}_2 - \vec{x}_1$, $\vec{b} = \vec{x}_4 - \vec{x}_3$ and $\vec{c} = \vec{x}_3 - \vec{x}_1$ [3]. Cranking out the cross product we obtain the first ratio given in Table 1. The distances for the other ratios in Table 1 are much easier to determine.

Type of ratio	Approx. Value	Exact Ratio
Tetrahedron edge length to rod diameter *d*	13.544	$\sqrt{2}\tau$
Internal tangent sphere diameter to rod diameter *d*	8.577	$\tau - 1$
Dodecahedron edge length to rod diameter *d*	5.919	τ/φ
Jig circle diameter to rod diameter *d*	16.294	$2\varphi\tau/\sqrt{2+\varphi}$
Jig tall tower height to rod diameter *d*	8.147	$\varphi\tau/\sqrt{2+\varphi}$
Jig short tower height to rod diameter *d*	5.035	$\tau/\sqrt{2+\varphi}$

Table 1. Ratios for the rod version, $\tau = \varphi^2\sqrt{15-\varphi} = \sqrt{27+40\varphi} \cong 9.577$.

For example, suppose we construct our puzzle from standard colored pencils with diameter $d = 7.2$mm, and $s = 14$mm diameter ball bearings. Let l be the rod length. The tetrahedron edge length is the distance between ball bearing centres ($l + s$), so Table 1 indicates we should cut our pencils at $l = (13.544)(7.2\text{mm}) - 14$mm or 83.5mm. This is the absolute minimum rod length — due to piece imperfections one may want to cut them slightly longer. In addition, we may want to inset the magnets into the rods, as in Figure 1a. To take this inset into account the rod length must be increased by 2mm (the exact amount is $s - \sqrt{s^2 - d^2}$). Ball bearings with diameter less than $s = \sqrt{3}d$ should not be used, because the fit of the puzzle will be off. For aesthetic reasons, somewhat larger ball bearings seem to be preferable (the Figure 1a puzzle is made from quarter-inch rods and 14mm ball bearings, so $s/d = 2.2$).

We can assemble the puzzle with a ball inside, and the largest that will fit has diameter 6.18cm = (8.577)(7.2mm). Stephen Chin has made a simple jig which can aid in assembly. His jig is five magnets embedded in a board at the vertices of a pentagon. The pentagon side length to be used for this is 4.26 cm. A more elaborate jig was invented by John Devost, he glues five short and five tall dowels to a board, equally spaced around a circle of diameter 11.7 cm

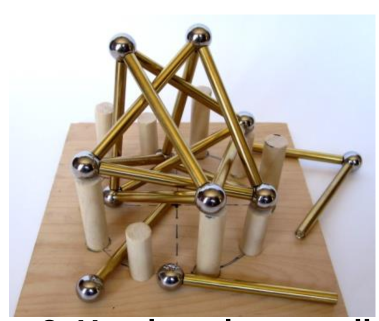

Figure 2. Hand-made tower jig.

(Figure 2). Using Table 1, we calculate that the tall and short towers have heights of 5.87 cm and 3.63 cm.

Lee Krasnow's Merkaba

Similar calculations can find the piece dimensions for this puzzle. The struts have a triangular cross section as shown in Figure 3a. The triangles are isosceles with the odd angle given by the dihedral angle of the tetrahedron, $\cos^{-1} 1/3 \cong 70.53°$, and the other two angles $\tan^{-1}\sqrt{2} \cong 54.74°$.

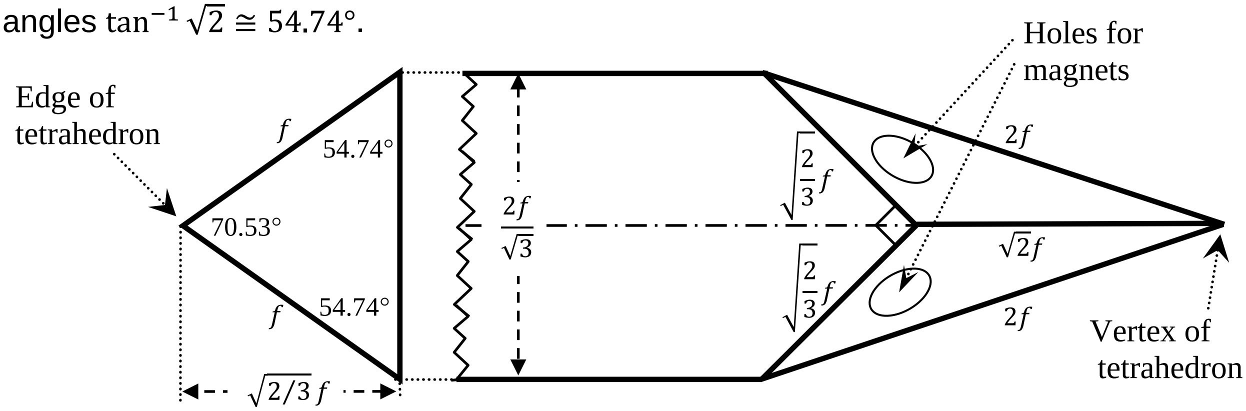

Figure 3. Cross-section of a strut, and bottom view of the end a strut.

We want to determine the value of *f* such that the struts just touch. In this case three struts touch at a point which is the midpoint of an edge of a tetrahedron. Conse-quently, this distance *f* can be calculated by another vector formula for the minimum distance between the midpoint $\vec{x}_0 = (0,0,1)$ (between $\vec{x}_1$and $\vec{x}_2$) and the edge from $\vec{x}_3 = R\vec{x}_1$ to $\vec{x}_4 = R\vec{x}_2$. This distance is $f = |(\vec{x}_3 - \vec{x}_0) \times (\vec{x}_4 - \vec{x}_0)|/|\vec{x}_4 - \vec{x}_3|$ [4], which gives the first ratio in Table 2. Note that the Lee Krasnow Merkaba of Figure 1b uses f=12.2 mm, which gives a tetrahedron edge length of 14.8 cm.

Type of ratio	Approx. Value	Exact Ratio
Strut length (tetrahedron edge length) to *f*	12.092	$8\varphi^2/\sqrt{3}$
Internal tangent sphere diameter to *f*	6.917	$2\varphi^3\sqrt{2}/\sqrt{3}$
Dodecahedron edge length to *f*	5.284	$4\varphi\sqrt{2}/\sqrt{3}$

Table 2. Ratios for the Merkaba.

Origami provides an inexpensive method of building this version [5, 6]. Here the struts are made from 1:3 paper rectangles which are folded twice along their long dimension. This gives a length to width ratio of 12, a good approximation to the Table 2 value of 12.092. Assembly of the paper version is not easy, some hints for this can be found on [5] (useful for all versions of this puzzle).

References

[1] John Devost, Puzzle Paradise, http://www.puzzleparadise.ca/
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