The Progressive Pyramid Puzzle

by George Bell

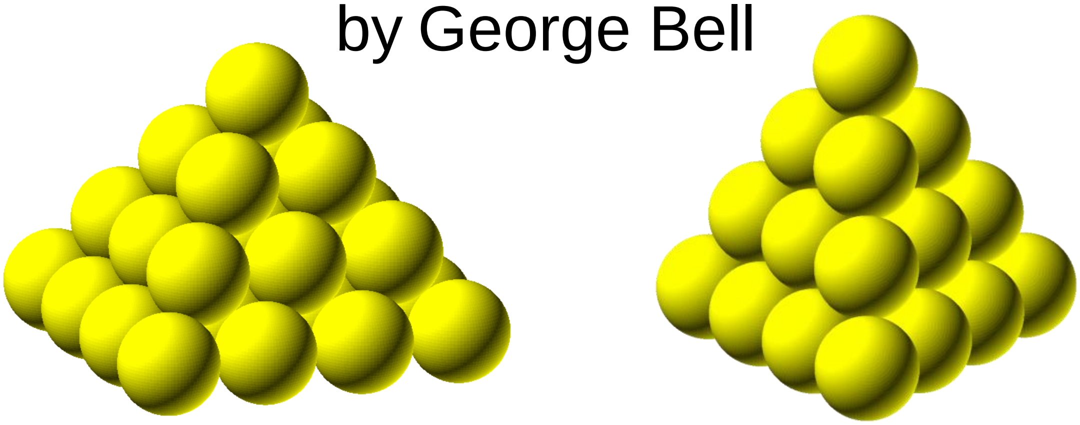

Figure 1. A 30-ball square-base pyramid, a 20-ball tetrahedron.

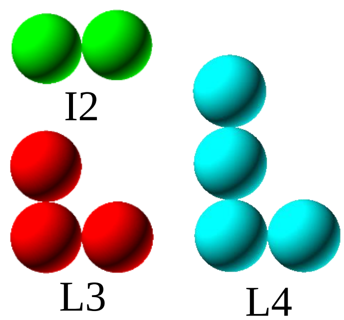

Figure 2. Basic puzzle pieces.

Ball pyramid puzzles have a 50-year history but they have never been among the more popular puzzles. This is good news for collectors because it means that a nice collection of ball pyramid puzzles can be obtained inexpensively.

One of the most well-known ball pyramid puzzles does not even have a name! The puzzle consists of 10 identical L3 pieces (10xL3), with the task to build a 30-ball square-base pyramid (Figure 1, left). This puzzle was described (but not named) by Leonard Gordon in 1986 [1]. It has been rediscovered by several designers, most notably by Torsten Sillke in 1992 [2]. Sillke called it *Die Verflixte Pyramide* [3] which google translates as *The Darn Pyramid*. I prefer the descriptive name: *Ten L Pyramid*.

At this point let us consider the other common shape for ball-pyramid puzzles: the tetrahedron (Figure 1, right). Can we build a tetrahedron (of any size) using only L3? The answer is no, and it is interesting to understand why. Consider any vertex ball of the tetrahedron. It is not hard to see that a vertex ball cannot be part of any L3. This is due to the fact that the vertex ball lies at the intersection of three hexagonal packing planes (three faces of the tetrahedron) while L3 lives in the "alternate universe" of orthogonal packing planes. We could say, in some sense, that the tetrahedron is "resistant" to being formed exclusively from L3's (or any planar piece with a 90° angle).

This is a story of a puzzle which was not created in a flash of inspiration, but grew in stages over the course of a year. Two years ago I wrote an article on hexagonal ball pyramids [4]. Figure 3 shows a hexagonal ball pyramid with 23 balls and 4 layers.

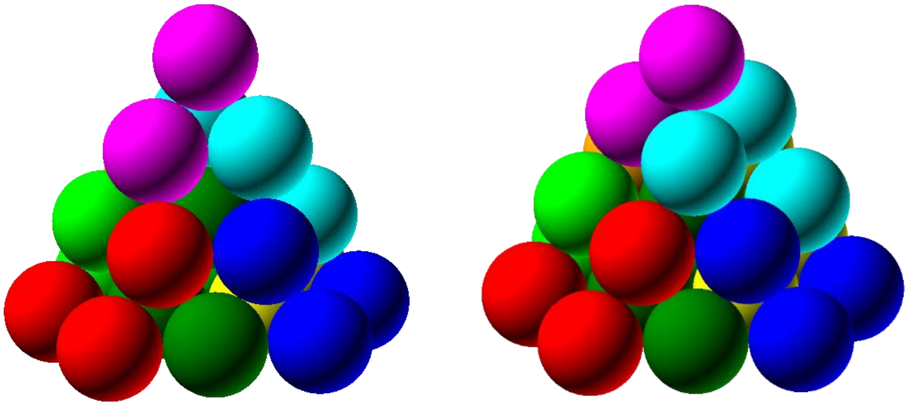

Figure 3. A 23-ball hexagonal pyramid (Seven L Pyramid), a twisted pyramid.

Clearly it is impossible to build a 23-ball hexagonal pyramid using only L3, because 23 is not a multiple of 3! In addition, the top two layers form a tetrahedron, so the top ball cannot be part of any L3. However, it was a surprise to discover this pyramid **can** be built from 7xL3 plus one I2. I made a copy of this puzzle, the *Seven L Pyramid*.

Playing with *Seven L Pyramid* I received another surprise: repositioning the top two pieces produces a different pyramid (Figure 3). I call this a "twisted pyramid" because it is a hexagonal pyramid with the top two layers rotated by 60°. The bottom three layers can also be considered three intersecting 10-ball tetrahedra. At the time, it seemed an undesirable alternate solution, but it is a hint of things to come.

I played around with the twisted pyramid shape and found that it can be made from I2, 3xL3 and 3xL4 (*Six L Pyramid*). Further experimentation with this piece set, I found I could build many other symmetric shapes. When the number of symmetric "pyramids" I could build reached 10, I realized I was looking at a multiple assembly puzzle.

Puzzle Geometry

Figure 4 shows three horizontal layering patterns for these pyramids. The colors represent symmetry partners which can be optionally removed, and the resulting "pyramid" retains 3-fold rotational symmetry. The "X" marks the center line of the resulting pyramid. Because pattern B is the only one with a ball in the center, if a pyramid is to end with one ball the top layer must be B.

Notice that there are no 90° angles between balls in Figure 4. Because of this, L3 and L4 can never lie flat in a layer, and it seems surprising that the Figure 3 pyramids can be made from only L3's (plus one I2).

A pyramid can be identified by the layering sequence (bottom to top) followed by the number of balls in each layer. For example, the pyramids in Figure 3 are identified as ABAB (12, 7, 3, 1) and ABCB (12, 7, 3, 1). The balls per layer sequence will be abbreviated as the "BPL sequence" or just "BPL".

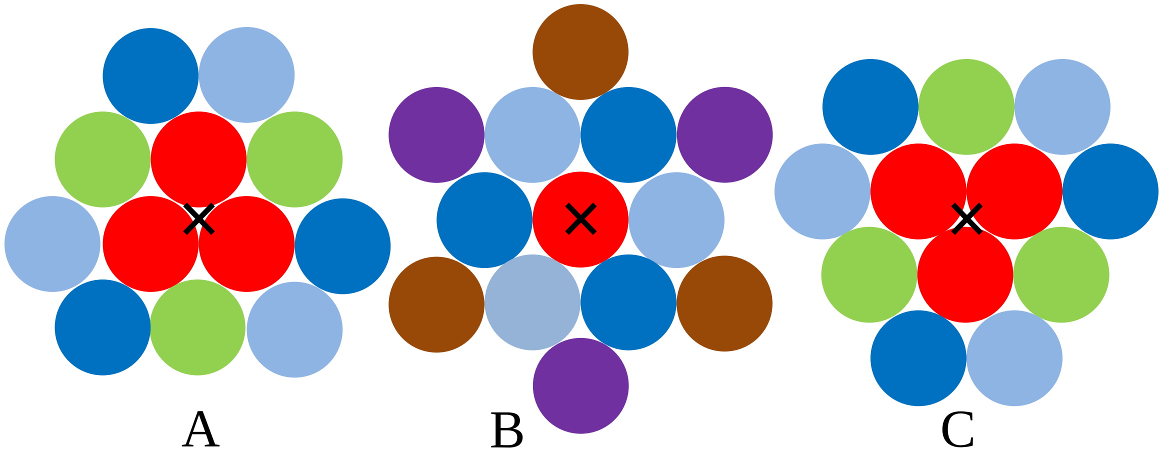

Figure 4. The three layering patterns (hexagonal packing layers).

We can add layers in any order, but to form a stable pyramid there are two "rules". First, the same layering pattern cannot be immediately repeated (the layering sequence cannot contain "AA", "BB", or "CC"). I jokingly refer to this as the "no ABBA" rule (my apologies to fans of the famous Swedish rock band!) Second, the BPL sequence must be decreasing. However, when designing as well as solving puzzles, it is always useful to "think outside the box", and consider if any interesting pyramids result from breaking these rules.

The main layering sequence used in almost all ball puzzles is ABC (repeating forever), or any combination of the three letters repeating forever (from a sphere-packing perspective, the labels A, B and C are arbitrary). This results in face-centred cubic (FCC) sphere packing. The Figure 1 tetrahedron in our notation is BCAB (10, 6, 3, 1).

Hexagonal close packing (HCP) [4] is defined by the layering sequence AB (repeating forever), or any sequence of two letters repeating forever. But there are layering sequences which are neither FCC or HCP, such as ABCB (Twisted Pyramid) and ACAB. One can also remove or add balls in a symmetrical fashion in each layer, producing a large number of shapes with triangular symmetry, all of which I call "pyramids".

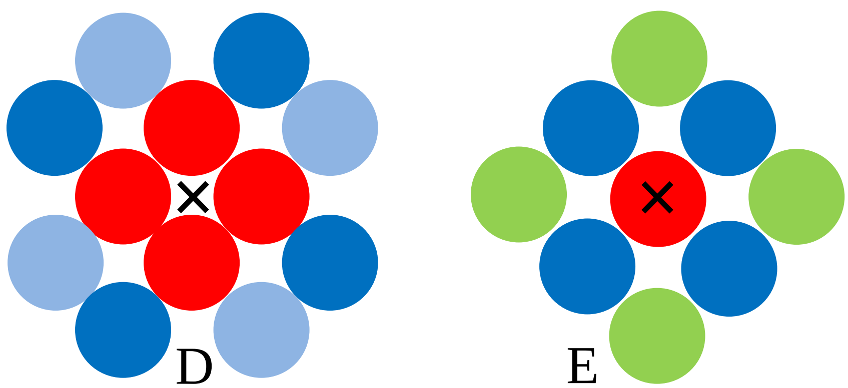

Figure 5. The two layering patterns (square packing layers).

Figure 5 shows the layering patterns possible for square packing layers. There are only two layering patterns (called D and E to distinguish them from the previous cases). The only possible "no ABBA" packing is DE (repeating forever), which rather surprisingly results in FCC packing. Many pyramids can be generated by removing balls in a symmetrical fashion, resulting in hollow pyramids or truncated pyramids. We could also start with a rectangle instead of a square. The *Ten L Pyramid* (Figure 1) in our notation is DEDE (16, 9, 4, 1).

The Progressive Pyramid

The idea behind *Progressive Pyramid* is a small set of pieces which can build as many possible symmetric shapes (pyramids) as possible. I played around with a number of piece sets. Clearly if one includes enough copies of I2 almost anything can be built, so all my piece sets used at most two copies of I2. Most interesting are pieces with 90° angles, and it seems that I2, L3 and L4 are the best options if there are only three piece types (Figure 2). After a lot of experimentation, the 7-piece set I settled on is: I2, 4xL3, 2xL4.

This piece set contains 22 balls, so it cannot build either pyramid in Figure 3. However, it turns out that this piece set can build five pyramids using all seven pieces, plus at least 20 other symmetric pyramids using a subset of the pieces.

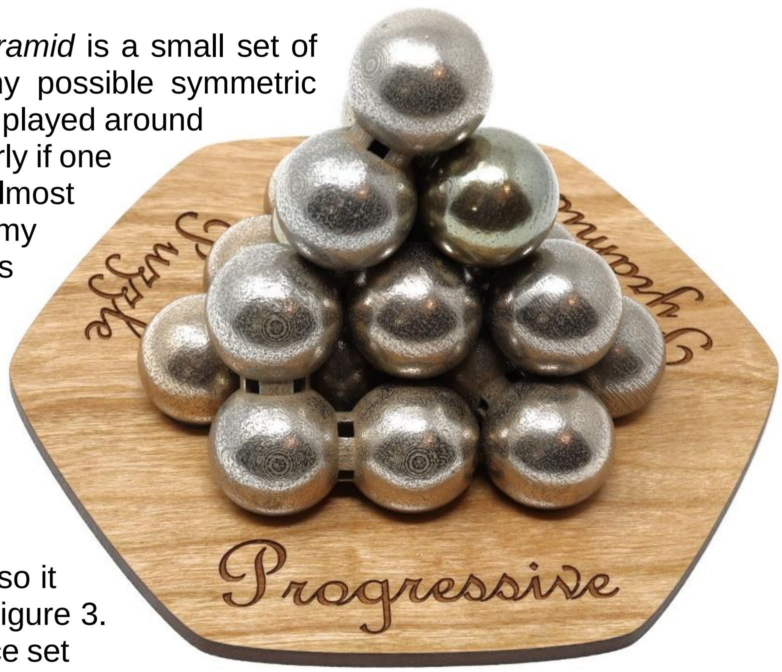

Figure 6. The *Progressive Pyramid,* 3D printed in nickel steel (showing the solution to challenge 14).

A wood version was my exchange puzzle for IPP39 in Kanazawa, Japan. I may have a few remaining copies for sale in my Etsy store [5].

Figure 7 and Table 1 give a set of 14 target shapes in roughly increasing difficulty. The final four puzzles have either a unique solution or two solutions which are variations of one another. The 19-ball octahedron has 23 solutions but many are not stable. All the puzzles in Table 1 can be solved by BurrTools [6], but those sphere packings which are not FCC require special techniques [4]. The only pyramids in Table 1 which require these special techniques are #11, #13 and #14.

I mentioned that 25 pyramids can be built, but Table 1 and Figure 7 show only 14. The remaining shapes are less interesting, in my opinion, and their discovery is left to the reader. I suggest three places to look for additional symmetric shapes:

1. Variations of the 3x3 square-based or 4x3 rectangular-based pyramids (hollow or truncated).
2. Variations on the 19-ball octahedron (hollow or truncated, many unstable).

3. Pyramids which violate the “no ABBA” rule. Easy examples are DD (4,4) and BB (7,7) (these two can be assembled at the same time from the seven *Progressive Pyramid* pieces). A dexterity challenge is BBBB (7,7,7,1), which uses all seven pieces but just falls apart.

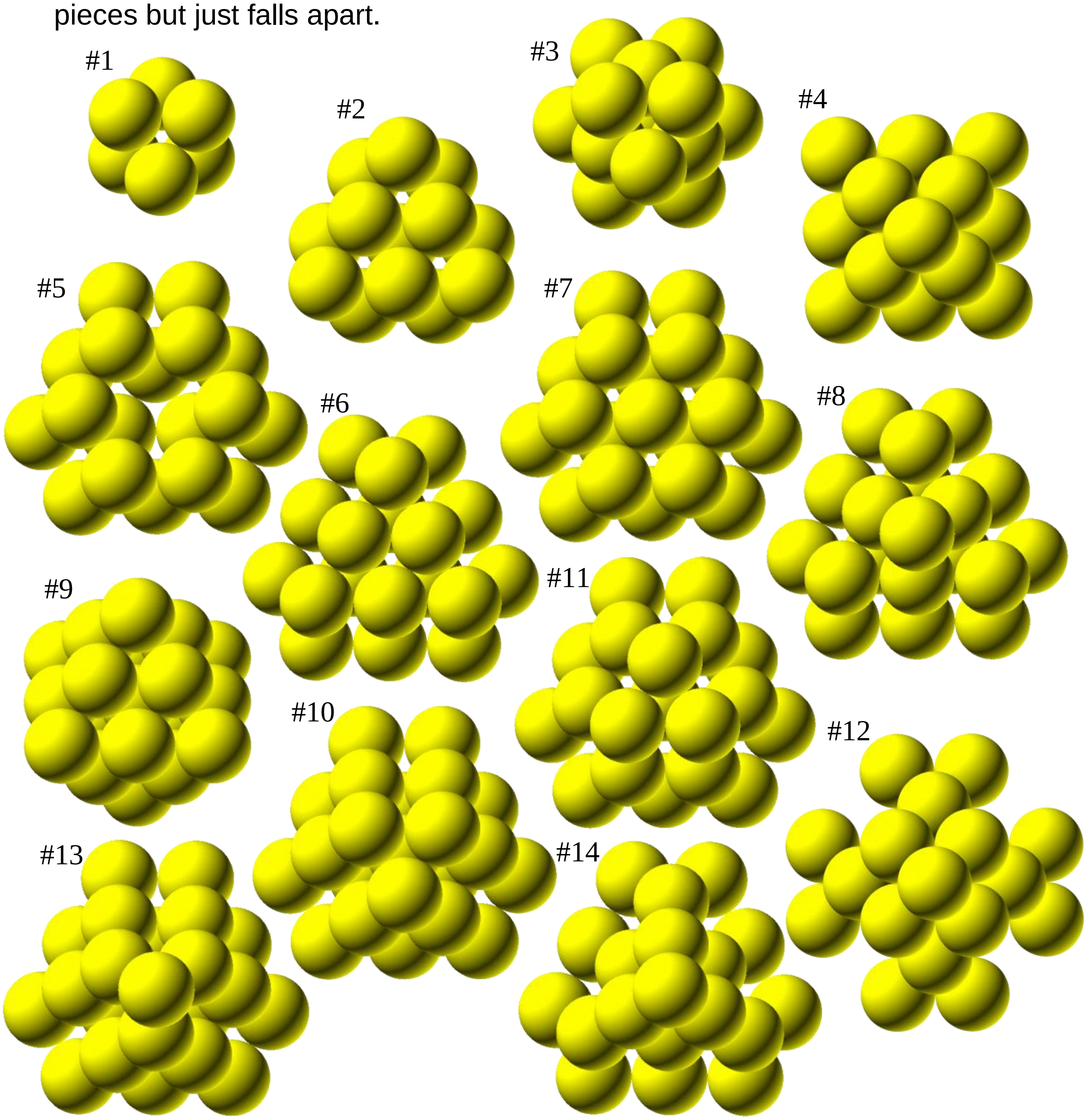

Figure 7. The 14 challenges of the *Progressive Pyramid* (top view).

Summary

Some puzzles were created in a flash of inspiration, but not *Progressive Pyramid*. The most elegant way to present this puzzle is with instructions: “Build a symmetric shape using some or all of the seven pieces.” This places the burden on the solver to come up with the shapes in Figure 7, and increases the difficulty. In order to eliminate large numbers of uninteresting (planar) solutions one should require that the resulting shape be at least two layers high.

A surprising number of symmetric pyramids can be built using these simple pieces, some are shown in Figure 7. But perhaps this is to be expected, for the sphere is the most symmetric of 3D solids.

#	Layers	Balls	Pcs.	Seq.	BPL	Solutions	Comments
1	2	6	2	AC	(3,3)	1	6-ball octahedron, easy
2	2	13	4	BC	(7,6)	2	easy
3	3	13	4	BAC	(7,3,3)	1	easy
4	3	14	4 or 5	EDE	(9,4,1)	6	square based pyramid, easy
5	2	18	6	AB	(12,6)	3	hollow, medium
6	2	18	6	AC	(12,6)	8	medium
7	2	19	6	AB	(12,7)	4	medium
8	3	19	6	ACB	(12,6,1)	8	medium
9	3	19	6	ABC	(6,7,6)	23	19-ball octahedron, difficult
10	3	22	7	ABC	(12,7,3)	54	medium
11	3	22	7	ABA	(12,7,3)	1	difficult, HCP
12	4	22	7	DEDE	(12,5,4,1)	2	difficult
13	4	22	7	ABCB	(12,6,3,1)	1	hollow twisted pyramid, difficult
14	4	22	7	ACAB	(12,6,3,1)	2	difficult

Table 1. The 14 challenges of *Progressive Pyramid*.

References

[1] Leonard Gordon, Some Notes on Ball-Pyramid and Related Puzzles, 1986. For a downloadable pdf, see http://www.gibell.net/puzzles/

[2] Torsten Sillke and Bernhard Wiezorke, Stacking Identical Polyspheres, Part 2: Square Pyramids, CFF 50 part 6, October 1999, pp. 14-18.

[3] Bernhard Wiezorke, Compendium of Polysphere Puzzles, 1996 (self-published).

[4] George Bell, Hexagonal Ball Pyramid Puzzles, CFF 106, July 2018, pp. 24-29.

[5] https://www.etsy.com/shop/PolyPuzzles

[6] BurrTools, http://burrtools.sourceforge.net/