The Dynamics of Spinning Polyominoes

George Bell   Mar 21st 2018   gibell@comcast.net

Q: Which two pentominoes are indistinguishable as rigid, rotating bodies?

A: Two rigid bodies have the same rotational dynamics if they have the same **principal moments of inertia**. Therefore, to answer this question, we need to calculate the **moment of inertia tensor** for each of the 12 pentominoes (the principal moments of inertia are the eigenvalues of this matrix). The pentominoes are 2D objects obtained by joining 5 squares along their edges in all possible ways. As spinning objects, we consider that each pentomino is composed of five 1x1 squares of mass 1 and thickness *h* as shown in Figure 1.

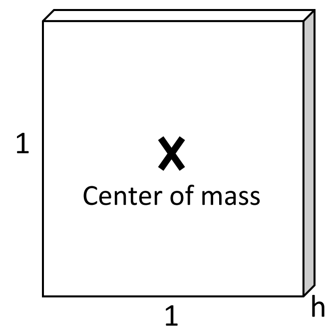

Figure 1: Polyomino building blocks.

As we will see, we can use any h between 0 and 1 and our results do not change qualitatively. We do not even have to build our pentominoes from squares of height *h*, we can use any object with square symmetry, or we could use circles or spheres (connected at points).

All moment of inertia tensors we will calculate are taken about the center of mass. The moment of inertia tensor J for an object composed of *n* squares is given by the sum of the moment of inertia tensors of the component squares, nJ_1, plus the moment of inertia tensor J_2 of the squares as point masses displaced from the center of gravity [1]. The moment of inertia tensor for the rectangular solid in Figure 1 is given by

$$J_1 = \frac{1}{12}\begin{pmatrix} 1+h^2 & 0 & 0 \\ 0 & 1+h^2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The limiting cases $h \to 0$ (thin plate) and $h = 1$ (cubes) are interesting special cases. In the cube case J_1 is 1/6 times the identity matrix.

Now we consider the contribution from the squares as point masses displaced from the center of gravity. As an example take the “P” pentomino (Figure 2).

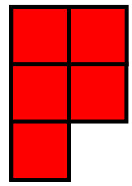

Figure 2: The P-pentomino.

The 5 squares have their centers at coordinates $(0,0), (0,1), (0,2), (1,1), (1,2)$. The center of mass of these 5 point masses is at $(0.4,1.2)$, so subtracting this from each coordinate we obtain a set of 5 point masses with center of mass at the origin:
$(-0.4,-1.2), (-0.4,-0.2), (-0.4, 0.8), (0.6,-0.2), (0.6,0.8)$. For a set of unit point masses in 2D at coordinates $(x_i, y_i, z_i = 0)$ the moment of inertia tensor about the origin is given by

$$J_2 = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

where

$$I_{xx} = \sum_i y_i^2 + z_i^2 = \sum_i y_i^2$$
$$I_{yy} = \sum_i x_i^2 + z_i^2 = \sum_i x_i^2$$
$$I_{zz} = \sum_i x_i^2 + y_i^2 = I_{xx} + I_{yy}$$
$$I_{xy} = -\sum_i x_i\, y_i$$

Calculating this for the P-pentomino, we get the total moment of inertia tensor about the center of gravity (0,0)

$$J = 5J_1 + J_2 = 5J_1 + \begin{pmatrix} 2.8 & -0.6 & 0 \\ -0.6 & 1.2 & 0 \\ 0 & 0 & 4.0 \end{pmatrix}$$

The principle moments of inertia are the eigenvalues of J, these are always real and non-negative. Because J_1 is diagonal, it only affects the magnitude of the eigenvalues. We calculate the eigenvalues of J_2 as 4, 3 and 1 with corresponding unit eigenvectors $(0,0,1)$, $(3,-1,0)/\sqrt{10}$ and $(1,3,0)/\sqrt{10}$. We now adopt the convention of displaying the pentomino with the two principal axes beginning at the center of mass, with length proportional to the magnitude of the eigenvector. The largest eigenvalue is always aligned with the z-axis and is not shown in these (2D) diagrams.

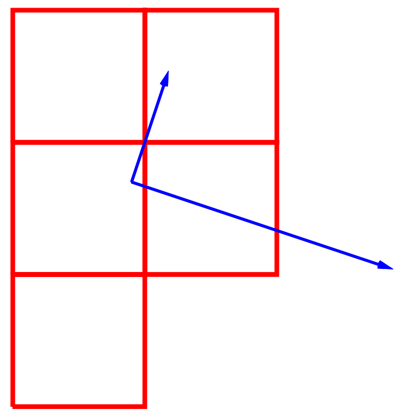

Figure 3: The P-pentomino, with principle axes of inertia shown at the center of mass.

We now repeat these calculations for all 12 pentominoes, with results shown in Table 1. This table shows the polyominoes sorted by decreasing principal moments of inertia. These eigenvalues are those of the matrix J_2, to obtain the eigenvalues of J we add $n/6$ to the largest eigenvalue and $n(1+h^2)/12$ to the other two. From here on out we assume $h = 0$ (2D polyominoes).

Name	λ_1	λ_2	λ_3
I	10	10	0
L	7.6	$(19+3\sqrt{29})/5 \cong 7.031099$	$(19-3\sqrt{29})/5 \cong 0.568901$
N	6.4	$(16+\sqrt{181})/5 \cong 5.890725$	$(16-\sqrt{181})/5 \cong 0.509275$
V	6.4	5	1.4
Z	6	$3+\sqrt{5} \cong 5.236068$	$3-\sqrt{5} \cong 0.763932$
Y	6	$3+\sqrt{5} \cong 5.236068$	$3-\sqrt{5} \cong 0.763932$
W	5.6	5	0.6
U	5.2	4	1.2
T	5.2	3.2	2
F	4.8	$(12+\sqrt{29})/5 \cong 3.477033$	$(12-\sqrt{29})/5 \cong 1.322967$
P	4	3	1
X	4	2	2

Table 1: The principal moments of inertia of the 12 pentominoes (same order as in Figure 4).

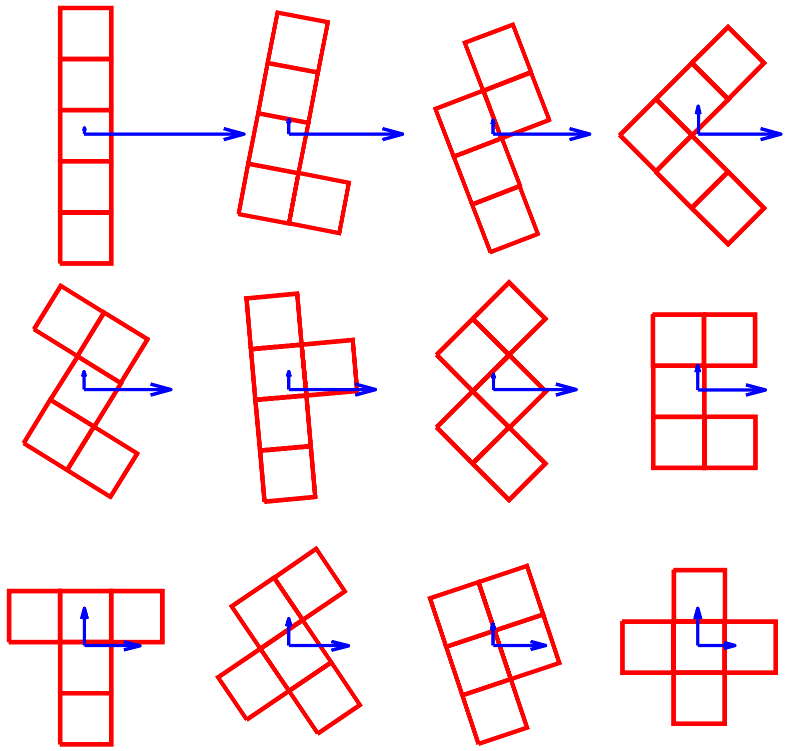

Figure 4: The 12 pentominoes oriented with principal axes aligned with the coordinate axes.

We see in Table 1 that there are exactly two pentominoes, Y and Z, which have identical principal moments of inertia. These will rotate exactly the same as freely spinning objects. If we rotate each so that their principal axes correspond, we obtain Figure 5a. We note that the Z pentomino has rotational symmetry, while the Y pentomino has no symmetry.

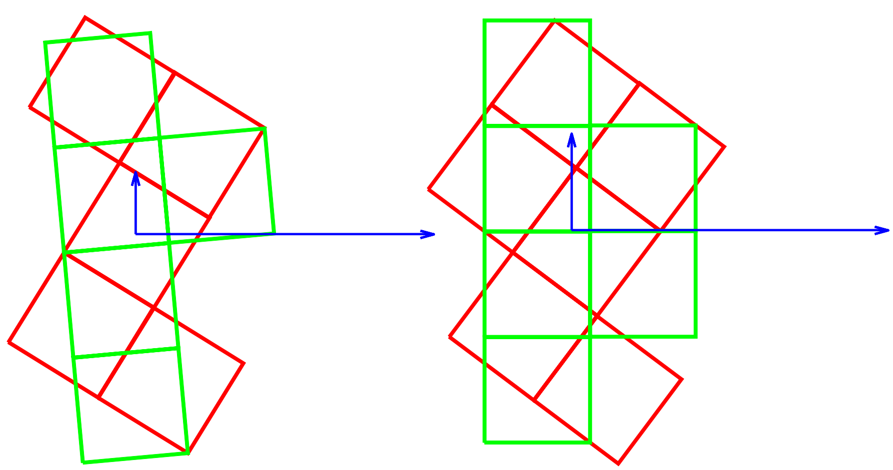

Figure 5: The Z and Y pentominoes with principle axes aligned and identical (left). The two hexominoes with identical moments of inertia.

What about larger polyominoes?

We have repeated these calculations through octominoes (n=8). There are exactly two hexominoes with the same principal moments of inertia (Figure 5b). These can be obtained from the Z and Y pentominoes by adding a square to each.

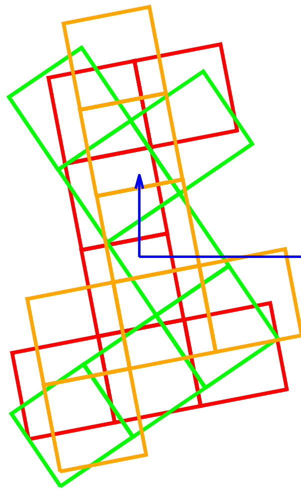

Figure 6: Three heptominoes (n=7) all sharing the same three principal moments of inertia.

Beyond n=6, polyominoes sharing the same principal moments of inertia are common. Figure 6 shows **three** septominoes (n=7) with the same principal moments of inertia. Figure 7 shows **six** octominoes which all share the same three principal moments of inertia: $(19.5,16,3.5)$!

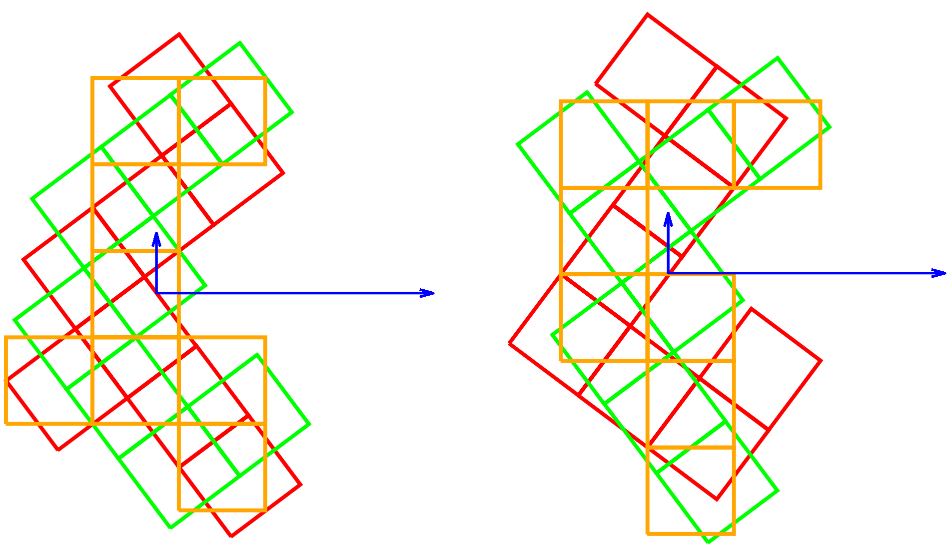

Figure 7: Six octominoes (n=8) all sharing the same principal moments of inertia.

[1] H. Goldstein, **Classical Mechanics**, Chapter 5, 1980 Addison-Wesley