Coordinate Motion Puzzles, Part 1

by George Bell

Coordinate motion (como) puzzles have always been among my favourites. Stewart Coffin was one of the originators of this technique. In his book, Geometric Puzzle Design, Coffin defines a coordinate motion puzzle as:

> A puzzle which, at some stage during the assembly, requires the simultaneous manipulation of three or more pieces or groups of pieces [1].

Coordinate motion has become fairly common in puzzles, and experts now look for it and know how to execute this subtle technique. In this article series we'll limit our scope to puzzles where **all** pieces participate in the coordinate motion, and all pieces move differently. We'll refer to this as **pure** coordinate motion. This eliminates many interesting puzzles, but otherwise the como category is just too broad.

This article series will appear in three parts which must be read simultaneously to make sense[1]. Part 1 will cover pure coordinate motion with 3 and 4-piece examples. Part 2 will deal with pure como puzzles with more than 4 pieces, and discuss visualization techniques. Designing como puzzles may appear difficult, as you can't construct them step by step and they only work perfectly or not at all. Part 3 will cover some techniques for designing como puzzles.

Puzzle geometry

In his book [1], Stewart Coffin presents the "toy" puzzle of Figure 1. This puzzle has 3 identical pieces (I use different colours to distinguish them). It can only be assembled by placing the pieces in the right configuration (Figure 1, right), and moving them together at the same time.

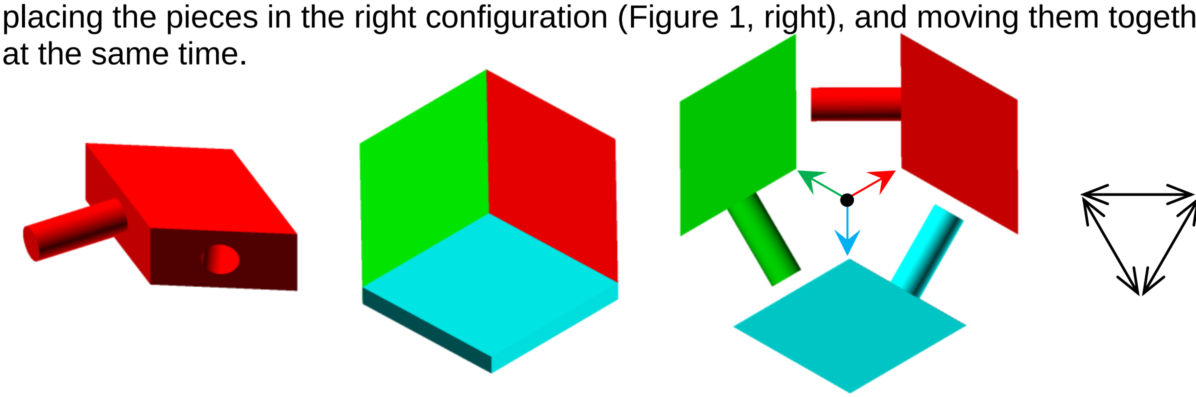

Figure 1. A single piece, the assembled puzzle, top view with PMV and RMV.

If we consider a single piece, it is the interaction with the other pieces which restricts its motion to a line. The rods and holes make this obvious, but in all pure como puzzles it is the interaction between the pieces which restricts the motion of each individual piece.

In this toy example each piece is restricted to move along a line, and all three pieces must move at the same time, or not at all. A mechanical engineer would say that this system has "1 degree of freedom"; such a como puzzle will be classified as "CM1".

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[1] Just kidding! I'm glad you are paying attention.

Although the majority of existing pure como puzzles are CM1, we'll see that other types are possible.

The piece movement vectors (PMV) show the linear directions followed by each piece as the puzzle comes apart. The PMV are key to understanding disassembly of these puzzles. If we take the differences between piece movement vectors we obtain the relative movement vectors (RMV) as described by Stewart Coffin [1], he refers to them as "vector diagrams". For the PMV, the colour of each arrow shows which piece it is associated with; for the RMV we use double-headed arrows to show v_1-v_2 and v_2-v_1 at the same time. In 2D, the RMV is a polygon, while for a 3D PMV, the RMV is a polyhedron.

The magnitude of each PMV gives the speed of each piece. If the interaction between the pieces is symmetric (as in the toy example) the speeds will be the same. But the interaction need not be symmetric, and then the pieces may move at different speeds. It can be surprisingly difficult to determine the PMV by observing a como puzzle coming apart. All 3-piece como puzzles share the PMV of the toy puzzle (Figure 1), we'll refer to it as P3, for planar and 3 pieces.

The toy example can be generalized to make a planar como puzzle with n pieces, for any n≥3. The associated PMV will be called Pn. If n is even the toy puzzle comes apart into identical halves, which can then be disassembled one piece at a time. By Coffin's definition these are not como puzzles, since their assembly doesn't **require** coordinate motion. I prefer to think of coordinate motion as a property of the solution, not the puzzle itself. Thus, I would say that the 4-piece toy puzzle has both a coordinate motion solution and a sequential solution.

A puzzle coming apart in halves is a common occurrence and it will be referred to as a "2-part solution". In general, if the pieces can be grouped into n parts, and each part moves as a unit, it will be referred to as an "n-part solution". If a puzzle has an n-part solution, and n=2, it is not a como puzzle (by Coffin's definition); if n>2, it is not a pure como puzzle (since pure como requires that all pieces move differently).

One important parameter for any como puzzle is the **disassembly length** of the coordinate motion before the pieces come apart. In the toy example this is the length of the rods. A long disassembly length is desirable, but is always limited by the geometry of the pieces.

Other types of pure coordinate motion puzzles

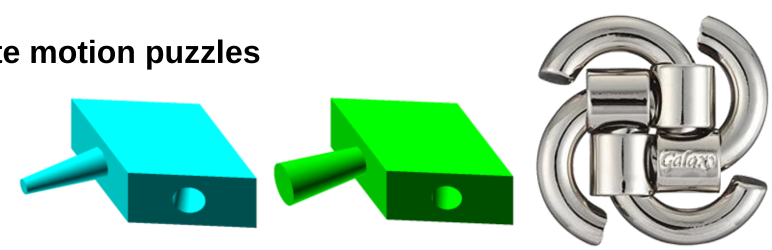

Figure 2. Artificial CM+ and CM0 pieces, *Cast Galaxy*.

In studying como puzzles I found a few which have more than one degree of freedom. Real examples will follow, but we can create a (somewhat artificial) example of this by tapering the rods (Figure 2). The aqua puzzle goes together using como, but seems loose. The tapered rods allow for piece translations which are not all in sync. This como puzzle has more than one degree of freedom and will be classified as “CM+”. If we can determine that the puzzle has exactly x degrees of freedom it will be called “CMx”.

The hallmark of a CM+ puzzle is this: take the assembled puzzle and move the pieces slightly apart. In this configuration you will notice that some of the pieces can move separately from the others. This is not the case for a CM1 puzzle, where the pieces must always move all together.

CM+ puzzles are basically loose como puzzles, so what? One reason is that it may be possible to “repair” them into CM1 puzzles. In the artificial example we can remove the taper in the rods, but we’ll see real examples shortly.

More surprising is the discovery of como puzzles with less than 1 degree of freedom. If the pieces have 0 degrees of freedom, they can’t move at all! Such bizarre objects exist, and we classify them as “CM0”. We can create such an object by tapering the rods and holes in the opposite direction (Figure 2, green). This produces a 3-piece object which can’t come apart, so it’s not even a puzzle (except maybe as an impossible object)!

The CM0 designation may sound pointless—it is useful only if we can get the como working again. In the examples to follow, we’ll see that these become the most interesting como puzzles of all. For when we “repair” a CM0 object, it may not become a CM1 puzzle, but something else entirely.

Up to this point, we have not mentioned piece rotations. We have considered coordinate motion starting from an assembled state, where the pieces move away from one another without rotation. If we look at this in reverse, the pieces move together until they collide, reaching a “hard stop” at the assembled state.

There exist pure como puzzles which don’t have a hard stop, always involving rotations. A good example is *Cast Galaxy* [3] by Bram Cohen. The solved state (Figure 2) is not a hard stop but rather a position where the 4 pieces are symmetrically arranged in a plane. As a mechanical linkage this puzzle has many degrees of freedom involving both piece translations and rotations. *Cast Galaxy* can’t be CM+, because it has no hard stop. We’ll call it “CMR”, for pure como with rotations and without a hard stop.

Three piece examples

Our first example is Stewart Coffin’s *Hole-In-One* [4]. The basic piece is shown in Figure 3. It can be made by gluing two Right-Handed Prism Blocks (R) to a Rhombic Pyramid Block (P). These are tetrahedral blocks as defined by Stewart Coffin [2]. This is the same dissection used in BurrTools [5] “Rhombic Tetrahedra” geometry.

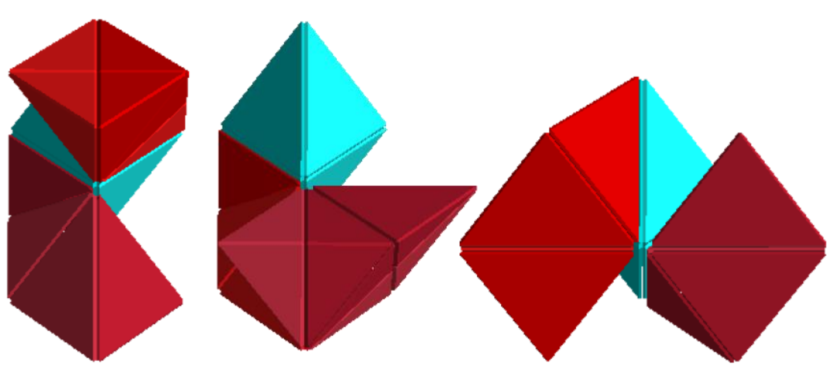

Figure 3. Hole-In-One base, Orchid Piece, Rose Piece. Each is made from a P (Aqua, fixed) and two R (Red).

If we put together three Hole-In-One base pieces, they form a rhombic dodecahedron (RD). We'll call this puzzle "*Loose Hole-In-One*", it is CM+. You can initiate the como by moving all three pieces away from the axis of symmetry (the z-axis). At this point each piece can move by itself slightly up or down (parallel to the axis of symmetry), the hallmark of CM+. There is a PMV where one piece moves away (from the z-axis), the second piece away and up, and the third away and down. In addition, this PMV can be rotated by 120°, resulting in 3 PMV's and (at least) 3 degrees of freedom (see [12]).

Coffin fixed *Loose Hole-In-One* by adding a hole to one piece and a peg to another (thus the name). This forces two pieces to move in a line directly toward or away from one another and results in a CM1 puzzle. It is this puzzle he calls "*Hole-In-One*" (STC #52A). The disassembly length is the length of the peg, which is relatively short.

This is not the only way to repair *Loose Hole-In-One*. I discovered that one can swap two of the Right-Handed Prism Blocks between two pieces. The resulting *Orchid Pennyhedron* [6] uses two Orchid pieces (Figure 3) and one Hole-In-One base.

Another repair option is to add to, or stellate the RD. The added rhombic tetrahedra must be connected to the three pieces, and with a little thought the resulting puzzle can be CM1. For the first stellation we need to add 12 P's. To make *Hole-In-None*, we add 4 P's to each *Hole-In-One* base piece, one of which is cut in half (green in Figure 4). *Hole-In-None* uses 1xA and 2xB, so named because this variation eliminates the hole. One can optionally remove the top and bottom P's in Figure 4, I call this version *Estrella* because it assembles into a star shape (same shape as in Figure 6).

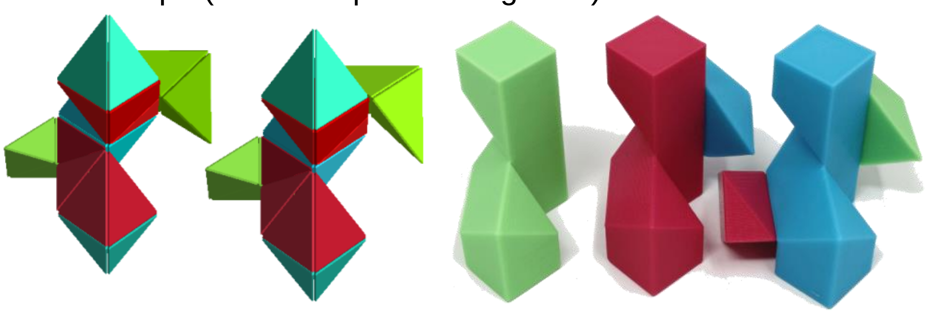

Figure 4. *Hole-In-None* pieces A and B, *Broken Burr* pieces.

Broken Burr is another variant where the P's are not cut in half, but 3, 4 and 5 P's are added to create three different pieces (Figure 4). The pieces are further extended so that the final shape appears to be a diagonal burr made from square stock. *Broken Burr* has four assemblies, all como and only one is symmetric. From the original *Loose Hole-In-One* have sprouted five variations.

The Rose piece (Figure 3) is also made by gluing two R's to a P. Three Rose pieces make an RD, but this puzzle (*Rose Pennyhedron* [6]) cannot be assembled when made from exact-sized, rigid pieces, it is CM0. As a mathematician it troubles me that I can't prove that this puzzle can't be assembled, but having made many I believe it to be true.

If one increases the offset a lot, *Rose Pennyhedron* does assemble. One can also force the pieces together—this stresses some interior edges, showing where they need to be rounded. After rounding these edges, the wood version goes together with a snap (the noise itself is an indication of CM0, no CM1 puzzle goes together with a snap).

Our next example is Stewart Coffin #118: *Three Bunnies* [4]. This puzzle consists of three different pieces—but one is special, we'll call it *Three Bunnies* base (Figure 5). This piece is special because it is the only piece where three copies form the final shape (3 RD's together). In fact, three copies of Three Bunnies base is CM+, it can come apart in multiple ways, like *Loose Hole-In-One*.

Coffin was able to repair the como by swapping 4 tetrahedral blocks between two of the pieces. This creates the 3 different pieces and forms the actual CM1 *Three Bunnies*. This is analogous to the process of going from *Loose Hole-In-One* to *Orchid*.

We consider now *Three Piece Separation* (STC #86A) [4]. This uses three identical pieces (Figure 5) and they come apart with a twist. I believe it is CM0. If you print the pieces with zero offset it goes together, but feels very tight. A copy with 0.1 mm offset feels loose. This puzzle is much looser than *Rose Pennyhedron*—but is still, I believe, CM0.

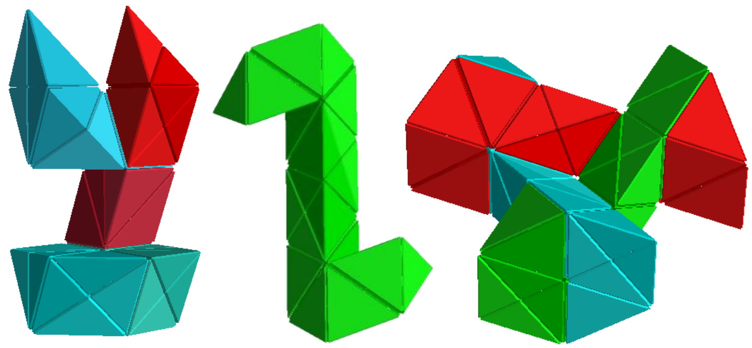

Figure 5. *Three Bunnies* base, made from 3 C (Cyan) and 2 R (Red), *Three Piece Separation* piece and assembled puzzle.

All these puzzles come apart in a single como movement. *CM13* [7] by Andreas Röver is different. This puzzle goes through a series of 4 como movements before coming apart. It is CM1, but uses two different PMV's in sequence (each rotated by 60°). Each piece moves in a zig-zag; this puzzle has the longest disassembly length of any como puzzle I know. At no point is any other piece movement possible, making the disassembly a simple maze without dead ends.

Four piece examples

There are several examples of 4-piece como puzzles using P4, the planar PMV. One is *Iwahiro's Apparently Impossible Cube* [8]*.* Another is what I call the *Vinco Core* [9], the 4-piece como cube which underlies all of his *New Range* series, such as *Bicone* [9]. All these lack a 2-part solution, despite having P4 as PMV. P4 only gives the piece movements when all 4 pieces move at the same time.

A second possible PMV for a 4-piece puzzle is T4, the pieces move outward through the vertices of a regular tetrahedron. The RMV is a regular tetrahedron. T4 is used by several puzzles, for example *De Doe Dak Ka* by Stephen Chin and Stuart Gee [6]. Four indentical pieces assemble via CM1 into an RD. This puzzle can have different outward appearances, it can be trimmed down to a cube and also hollowed out [6]. T4 is also the PMV for Vinco's *Ikeburra* (IPP28, 2008), and Craighill's *Tetra Puzzle* [10, 11].

De Doe Dak Ka uses 4 copies of piece B in Figure 6. Five other puzzles can be obtained using combinations of B, A and M, these are given in the supplemental material for this CFF article [12]. *Frankenstar* uses 2 B pieces plus 2 A pieces and I stellated the RD, making all 4 pieces different. This puzzle has 9 assemblies, all como and only one is symmetric.

Most of these puzzles are CM1, the exception is 2 A pieces plus 2 M pieces, this combination has a fascinating CM2 solution. The four pieces can come apart via T4 or T4 rotated by 90°, or any linear combination of the two (see [12]). I noticed two pieces which touch along a face, and both como movements are parallel to this face. To create *Maze Pennyhedron*, I added a pin to one face and a maze in the other (Figure 6). A path through a 2D maze is 1D, therefore this maze has the effect of removing a degree of freedom, making *Maze Pennyhedron* CM1. We only need one maze, because all moves through it are coordinate motion. However, I found that the puzzle is more stable with a pin and maze in each piece. This is certainly the most unusual "repair" of a CM+ puzzle.

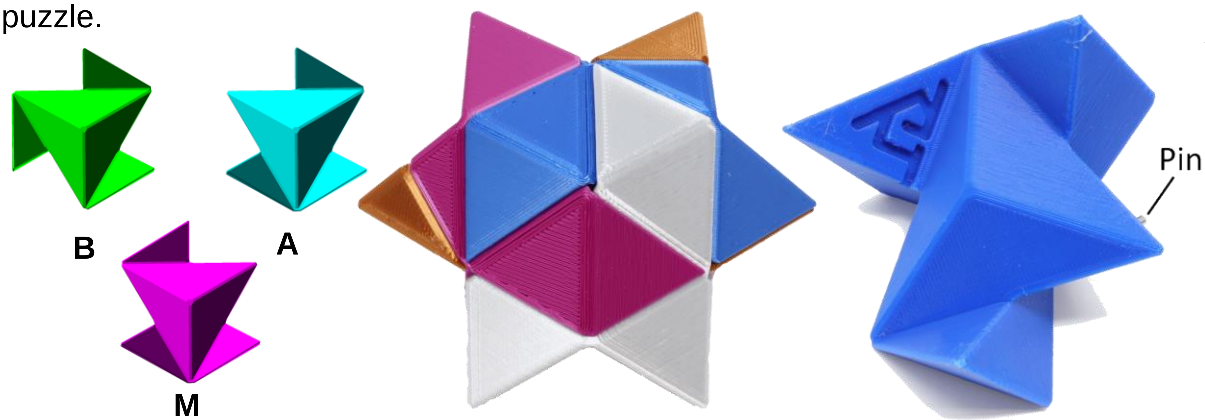

Figure 6. Pieces (B, A, M). *Frankenstar*, *Maze Pennyhedron* piece.

Finally, a few Coffin 4-piece puzzles deserve mention. *Fourth Dimension* (STC #94) has two como assemblies, both use T4 as PMV. Three puzzles use 4 rods with circular notches (STC #245, #246 and #X-70). These three puzzles are examples of CMR (pure como without a hard stop involving rotations).

Into the future

I believe the majority of coordinate motion puzzles are yet to be discovered. Considering the variety of 3 and 4-piece como puzzles, I think it is clear we have barely scratched the surface on the subject. Next, we'll move on to pure como puzzles with more than 4 pieces.

References

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